

On the solutions of the minimum energy problem in one dimensional sensor networks

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Abstract

We discuss solutions of the minimum energy problem in one dimensional wireless sensor networks for the data transmission cost function $E(x_i, x_j) = d(x_i, x_j)^a + \lambda d(x_i, x_j)^b$ with any exponent $a, b \in \mathbb{R}$ and $\lambda \geq 0$, where $d(x_i, x_j)$ is a distance between transmitter and receiver. We define the minimum energy problem in terms of sensors signal power, transmission time and capacities of a transmission channels. We prove, that for the point-to-point data transmission method utilized by the sensors in the physical layer, when the transmitter adjust the power of its radio signal to the distance to the receiver, the solutions of the minimum energy problem written in terms of data transmission cost function and in terms of the sensors signal power coincide.

Key words: sensor network, energy management, channel capacity.

1 Introduction

Characteristic feature of sensor networks is that these consist of small electronic devices with limited power and computational resources. Typical activity of sensor network nodes is collection of sensed data, performing simple computational tasks and transmission of the resulting data to a fixed set of data collectors. The sensors utilize most of their energy in the process of data transmission, this energy grows with the size of the network and the amount of data transmitted over the network. Generally in sensor networks there are two models of energy consumption. The objective of the first model is to maximize the functional lifetime of the network, [1, 2]. For this type of problems the data transmission in the network is modeled in such a way, that the energy consumed by each sensor is minimal. To extend the network lifetime the sensors must share their resources and cooperate in the process of data transmission. For typical solutions of such problems the consumed energy is evenly distributed over all nodes of the network. The second type of problems is to optimize the energy consumed by the whole network, [3, 4, 5, 6, 7]. Such problems arise when the network nodes are powered by the central source of energy or the node batteries can be recharged and the total energy consumed by the network is to be minimized. To solve this type of problems it is enough to find the optimal energy consumption model of each sensor and summed up their energies. In this paper we discuss solutions of the minimum energy problem in one dimensional wireless sensor network S_N when each sensor generates the amount Q_i of data and sends it, possible via other sensors, to the data collector. We assume, that the network S_N is build of N sensors and one data collector. The sensors are located at the points $x_i > 0$ of the line and the data collector at the point $x_0 = 0$. We prove, that for any exponent $a \in \mathbb{R}$ in the data transmission cost function

$$E(x_i, x_j) = |x_i - x_j|^a, \quad (1)$$

where $|x_i - x_j|$ is a distance between transmitter and receiver, when in the interval $(x_0, \frac{1}{2}x_N)$ there are N' sensors, there are $(N' + 1)$ solutions of the problem. We show, how to determine the optimal solutions of the minimum energy problem for the data transmission cost function which is sum of two factors

$$E(x_i, x_j) = |x_i - x_j|^a + \lambda |x_i - x_j|^b, \quad (2)$$

where $a, b \in \mathbb{R}$ and $\lambda \geq 0$. We also discuss the solutions of the minimum energy problem in wireless sensor networks when the energy utilized by the network is expressed in terms of node signal power, transmission time and there are constraint on the transmission channels due to presence of the noise and interference, [8].

We represent a sensor network S_N as a directed, weighted graph $G_N = \{S_N, V, E\}$ in which S_N is a set of graph nodes, V is the set of edges and E set of weights. Each directed edge $T_{i,j} \in V$ defines a communication link between i -th and j -th node of the network. To each edge $T_{i,j}$ we assign a weight $E(x_i, x_j) \equiv E_{i,j}$, which is the cost of transmission of one unit of data between i -th and j -th node. The data flow matrix $q_{i,j}$ defines the amount of data transmitted along the edge $T_{i,j}$. By $U_i^{(\text{out})} \subseteq S_N$ we denote a set of the network nodes to which the i -th node can send the data, i.e., $U_i^{(\text{out})} = \{j \in S_N \mid \exists T_{i,j} \in V\}$. The set $U_i^{(\text{out})}$ defines the maximal transmission range of the i -th node. In the paper we assume, that each sensor of S_N can send the data to any other node of the network $\forall i \in S_N \ U_i^{(\text{out})} = S_N$. If we assume, that each sensor of the S_N network generates the amount Q_i of data, $i \in [1, N]$, and the data is sent to the data collector, then the energy consumed by the i -th sensor in the process of data transmission can be written in the form $E_i(q) = \sum_{j=1}^N q_{i,j} E_{i,j}$. For the total energy consumed by the network $E_T(q) = \sum_{i \in S_N} E_i(q)$ the minimum energy problem can be defined by the set of following formulas

$$\begin{cases} \min_q E_T(q), \\ \sum_i q_{i,j} = Q_i + \sum_j q_{j,i}, \\ E_{i,j} \geq 0, q_{i,j} \geq 0, Q_i > 0 \ i, j \in [0, N], \end{cases} \quad (3)$$

where the second formula defines the feasible set of the problem. It states that the amount of data generated by the i -th node Q_i and the amount of data received from other nodes $\sum_j q_{j,i}$ must be equal to the amount of data which the node can send $\sum_j q_{i,j}$.

Because the objective function $E_T(q)$ of the problem is continuous and linear we can deduce from this a simple but helpful fact, that any local minimum of $E_T(q)$ is a global one and thus it is a solution of (3).

If we assume, that we search for a solution of (3) in the integers, i.e. $q_{i,j} \in \mathbb{Z}_+^0$ for $Q_i \in \mathbb{Z}_+$, then we get the mixed integer linear programming problem. It is easy to see that such problem is NP-hard. To find the minimum of $E_T(q)$ first we must find an integer matrix q satisfying the feasible set equation, given by the second relation in (3). Because this requires solution of the partition problem, [9], we get the reduction of the partition problem to the minimum energy problem (3) with the requirements $q_{i,j} \in \mathbb{Z}_+^0$, $Q_i \in \mathbb{Z}_+$.

2 Solution of the problem with the monomial cost function

In this section we solve the minimum energy problem (3) for the data transmission cost function (1) with arbitrary real value of the exponent a . As can be seen, the monomial (1) for $a \geq 1$ and $x_i \geq x_j \geq x_k \geq 0$ satisfies the inequality

$$|x_i - x_j|^a + |x_j - x_k|^a \leq |x_i - x_k|^a, \quad (4)$$

and it is an example of a super-additive function, [10]. This is because for $x_i - x_j = x$, $x_j - x_k = y$ (4) can be written in the form $|x|^a + |y|^a \leq |x + y|^a$. Solutions of the minimum energy problem

in S_N with the cost function (1), where $a \geq 1$, can be easily generalized to any data transmission cost function $E(x_i, x_j)$ which satisfies the inequality

$$\forall x_i \geq x_j \geq x_k \geq 0 \quad E(x_i, x_j) + E(x_j, x_k) \leq E(x_i, x_k). \quad (5)$$

From (5) it follows that the energy consumed by each sensor is minimal when it sends all of its data to the nearest neighbor in the direction of data collector. Let us assume, that the data is transmitted between two nodes located at the points x_i and x_k and (5) is satisfied, then the cost of transmission $E(x_i, x_k)$ can be reduced by transmitting the data via the j -th node located between them, i.e., via the point x_j for which the inequality is satisfied $x_i \geq x_j \geq x_k \geq 0$. Because the total energy consumed by the network is a sum of energies consumed by its nodes, then the solution of the minimum energy problem (3) with (1) and $a \geq 1$ can be described by the transmission graph $T^{(0)} = \{T_{i,i-1}^{(0)}\}_{i=1}^N$, with the weight of each edge $T_{i,i-1}^{(0)}$ equal to $q_{i,i-1}^{(0)} = \sum_{j=i}^N Q_j$. The graph $T^{(0)}$ defines the next hop data transmission along the shortest path, where the shortest path means transmission along the distance $d(x_i, x_j)$ between the transmitter and the receiver.

For $a \leq 1$ elements of the data transmission cost function (1) are the sub-additive functions, i.e., satisfy the inequality $|x|^a + |y|^a \geq |x + y|^a$, [10]. Solutions of the minimum energy problem (3) with (1) and $a \leq 1$ can be generalized to the data transmission cost function $E(x_i, x_j)$ which satisfy the inequality

$$E(x_i, x_j) + E(x_j, x_k) \geq E(x_i, x_k). \quad (6)$$

The optimal behavior of the sensors which minimizes the total network energy $E_T(q)$ can be deduced from the inequality (6), but it does not uniquely determine the solution of (3). To get the unique solution of (3) we need a concrete form of the data transmission cost function, for example (1) or (2). From (6) it follows that the cost of transmission between two nodes located at the points x_i and x_j is minimal when the data is transmitted along the longest hops, i.e., any transmission via node which lie between x_i and x_j is less optimal. For the data transmission cost function (1) and $a \in (-\infty, 1]$ one may expect, that the optimal data transmission is given by the graph $T^{(1)} = \{T_{i,0}^{(1)}\}_{i=1}^N$, with the weights $q_{i,0}^{(1)} = Q_i$. This is true for sensors which lie in the interval $[x_N, \frac{1}{2}x_N]$ of S_N . When $x_i \in [x_N, \frac{1}{2}x_N]$, then the distance $d(x_i, x_0)$ between the transmitter and the data collector is maximal and the inequality (6) for $x_i, x_k = 0$ and any sensor $j \in S_N$, i.e., not only for $x_j < x_i$ but also for $x_j > x_i$, is satisfied. For sensors which lie in the interval $(0, \frac{1}{2}x_N)$ to find the optimal transmission it must be taken into account two data transmission paths to the data collector. The directly to the data collector transmission path $T_i^{(1)} = \{T_{i,0}^{(1)}\}$ and the two hops transmission given by the path $T_i^{(1')} = \{T_{i,N}^{(1')}, T_{N,0}^{(1')}\}$. Selection of which one depends on the value of the parameter $a \in (-\infty, 1]$ in (1). Let us assume, that in the interval $(0, \frac{1}{2}x_N)$ there are N' sensors. For each sensor k from the interval $(0, \frac{1}{2}x_N)$ we split the network S_N into two sets $V_1^{(k)}$ and $V_2^{(k)}$. To the set

$$V_1^{(k)} = \{i \in S_N \mid d(x_i, x_0) < d(x_k, x_0)\}, \quad k \in [1, N']$$

belong sensors which lie to the left the k -th sensor. The set $V_2^{(k)}$ is the completion of $V_1^{(k)}$, i.e., $V_2^{(k)} = S_N \setminus V_1^{(k)}$. The sensors from the interval $(0, \frac{1}{2}x_N)$ can be used to classify solutions of the minimum energy problem for the data transmission cost function (1) and any value of the exponent $a \in (-\infty, 1]$. For the k -th sensor from the interval $(0, \frac{1}{2}x_N)$, we must check whether the optimal data transmission path from the k -th sensor to the data collector is $\{T_{k,0}^{(k)}\}$ or $\{T_{k,N}^{(k)}, T_{N,0}^{(k)}\}$. In other words, we must check the values of the parameter a for which the inequality holds

$$E(x_k, x_N) + E(x_N, x_0) \geq E(x_k, x_0), \quad k \in [1, N'].$$

Instead solving these inequalities, we solve the set of equations

$$E(x_k, x_N) + E(x_N, x_0) - E(x_k, x_0) = 0, \quad k \in [1, N'],$$

which for (1) have the form

$$|x_N - x_k|^a + x_N^a - x_k^a = 0, \quad a \in (-\infty, 1], \quad k \in [1, N']. \quad (7)$$

For N' roots a_k of (7) we can form N' intervals $a \in [a_{k+1}, a_k]$, $k \in [0, N']$, where $a_0 = 1$ and $a_{N'+1} = -\infty$. For any $a \in [a_{k+1}, a_k]$ the following set of inequality holds

$$\begin{cases} |x_N - x_k|^a + x_N^a - x_k^a \leq 0, \\ |x_N - x_{k+1}|^a + x_N^a - x_{k+1}^a \geq 0, \end{cases} \quad (8)$$

which means, that for $a \in [a_{k+1}, a_k]$ the nodes $i \in [1, k]$ transmit data along the two hops path $\{T_{i,N}^{(k+1)}, T_{N,0}^{(k+1)}\}$ and the nodes $i \in [k+1, N]$ along the one hop path $\{T_{i,0}^{(k+1)}\}$. The above results summarizes the following

Lemma 1. The solutions of the minimum network energy problem for the data transmission cost matrix $E_{i,j} = |x_i - x_j|^a$ and $a \in R$ is given by the data transmission graphs

$$\begin{cases} T^{(0)} = \{T_{i,i-1}^{(0)}\}_{i=1}^N & \text{for } a \in [1, \infty), \\ T^{(1)} = \{T_{i,0}^{(1)}\}_{i=1}^N & \text{for } a \in [a_1, 1], \\ T^{(k+1)} = \{T_{i,N}^{(k+1)}, T_{i',0}^{(k+1)}\}_{i=1, i'=k+1}^{k,N} & \text{for } a \in [a_{k+1}, a_k], k \in [1, N'-1], \\ T^{(N'+1)} = \{T_{i,N}^{(N'+1)}, T_{i',0}^{(N'+1)}\}_{i=1, i'=N'+1}^{N',N} & \text{for } a \in (-\infty, a_{N'}], \end{cases} \quad (9)$$

with the weights

$$\begin{cases} q_{i,i-1}^{(0)} = \sum_{j=i}^N Q_j, \quad i \in [1, N], \\ q_{i,0}^{(1)} = Q_i, \quad i \in [1, N], \\ q_{i,N}^{(k+1)} = Q_i, \quad i \in [1, k], \quad q_{i',0}^{(k+1)} = Q_{i'}, \quad i' \in [k+1, N-1], \\ q_{N,0}^{(k+1)} = Q_N + \sum_{j=1}^k Q_j, \quad k \in [1, N'], \end{cases} \quad (10)$$

where a_k are roots of the equations (7).

Detailed proof of the Lemma 1 can be found in [11]. On Figure 1 it is shown the optimal data transmission graph for the minimum energy problem when $a \in [a_{k+1}, a_k]$ and $k \in [1, N']$.

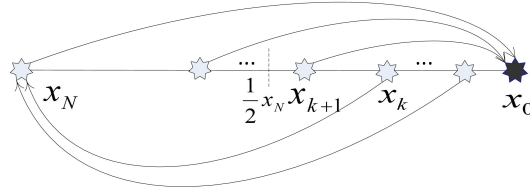


Figure 1: Optimal data transmission in S_N for $a \in [a_{k+1}, a_k]$.

The data transmission cost function

$$E_{i,j}(\bar{\lambda}, \bar{\alpha}) = \sum_n \lambda_n |x_i - x_j|^{\alpha_n}, \quad \forall_n \lambda_n \geq 0, \quad (11)$$

which is a sum of monomials (1) with nonnegative coefficients λ_n , for $\forall_n \alpha_n \geq 1$ it satisfies (5) and for $\forall_n \alpha_n \leq 1$ (6). The following lemma defines conditions under which the solution of (3) for the data transmission cost function (11) is given by (9).

Lemma 2. For the data transmission cost function (11), where $\forall_n \alpha_n \in [a_k, a_{k-1}]$, $k \in [1, N']$ and $\lambda_n \geq 0$ the solution of the minimum energy problem is given by the weighted transmission graph (9).

Proof. We can split the total energy consumed by the network into summands such, that $E_T(q) = \sum_n \lambda_n E_T^n(q)$, where $E_T^n(q) = \sum_{i \in S_N} E_i(q, E^n)$ and $E_{i,j}^n = |x_i - x_j|^{\alpha_n}$. Because for each $\alpha_n \in [a_k, a_{k-1}]$ the minimal energy utilized by the sensors is given by the same transmission graph $T^{(k)}$ from (9), then the optimal data transmission for $E_{i,j}(\bar{\lambda}, \bar{\alpha})$ is also given by $T^{(k)}$. \diamond

3 Solution of the problem with the polynomial cost function

We note that, the objective function $E_T(q)$ is linear and continuous and for this reason its minima lie on the border of the feasible set defined by the second relation in (3). From this follows, that the data transmitted by each node cannot be split and it must be sent to a single receiver. From the Lemma 2 we know, that when the exponents a, b in the data transmission cost function (2) belong to the same interval $[a_k, a_{k-1}]$, then the optimal transmission graph for $E_{i,j} = |x_i - x_j|^a$ and $E_{i,j} = |x_i - x_j|^b$ is the same and it is also optimal for $E_{i,j} = |x_i - x_j|^a + \lambda |x_i - x_j|^b$. The problem arises when the solutions of (3) for two exponents a and b in $E_{i,j} = |x_i - x_j|^a$ are given by different data transmission graphs T_a and T_b . This is because, the optimal transmission graph for the cost function (2) cannot be sum of the graphs T_a and T_b , unless the transmitted data in the graph $T_a \cup T_b$ are not split.

In this section we consider solutions of (3) in the one dimensional, regular sensor network L_N for which the nodes are located at the $x_i = i$ points of the half line. For the L_N network the data transmission cost matrix (2) has the form

$$E_{i,j} = |i - j|^a + \lambda |i - j|^b. \quad (12)$$

All presented below results are also valid for the non-regular network S_N , but the formulas are cumbersome because of their size and they will not be presented here.

The following lemma describes solution of the the minimum energy problem for the L_N network when the exponents a and b in (12) belong to the neighboring intervals $[a_k, a_{k-1}]$.

Lemma 3. For the data transmission cost matrix (12), when

$$\begin{cases} a \in [1, \infty), & b \in [a_1, 1], \\ a \in [a_k, a_{k-1}], & b \in [a_{k+1}, a_k], \quad k \in [1, N' - 1], \\ a \in [a_{N'}, a_{N'-1}], & b \in (-\infty, a_{N'}], \end{cases}$$

the solution of the minimum energy problem is given by

$$\begin{cases} T^{(0)} \text{ for } \lambda \in [0, \lambda_0], & T^{(1)} \text{ for } \lambda \in [\lambda'_0, \infty), \\ T^{(k)} \text{ for } \lambda \in [0, \lambda_k], & T^{(k+1)} \text{ for } \lambda \in [\lambda_k, \infty), \quad k \in [1, N' - 1], \\ T^{(N')} \text{ for } \lambda \in [0, \lambda_{N'}], & T^{(N'+1)} \text{ for } \lambda \in [\lambda_{N'}, \infty), \end{cases} \quad (13)$$

where

$$\begin{cases} \lambda_0 = \frac{2^a - 2}{2 - 2^b}, \\ \lambda'_0 = \frac{N^a - |N - N' - 1|^{a - (N' + 1)^a}}{|N - N' - 1|^b + (N' + 1)^b - N^b}, \\ \lambda_k = \frac{(N - k)^a + N^a - k^a}{k^b - (N - k)^b - N^b}, \quad k \in [1, N'], \end{cases} \quad (14)$$

$a_0 = 1$, $N' = \frac{N-2}{2}$ for N even and $N' = \frac{N-1}{2}$ for N odd, $T^{(k)}$, $k \in [0, N']$ is a set of data transmission graphs given by (9).

Proof. If $a \in [a_k, a_{k-1}]$ and $b \in [a_{k+1}, a_k]$ in (12), then we know from Lemma 1 that for a sufficiently small λ the solution of (3) is given by the weighted transmission graph $T^{(k)} = \{T_{i,N}^{(k)}, T_{i',0}^{(k)}\}_{i=1, i'=k}^{k-1, N}$, and for a sufficiently large λ the solution is given by $T^{(k+1)} = \{T_{i,N}^{(k+1)}, T_{i',0}^{(k+1)}\}_{i=1, i'=k+1}^{k, N}$, $k \in [2, N']$. Because of the linearity and continuity of the objective function $E_T(q)$ its minimum lies on the border of the feasible set. This means that the data transmitted by each node cannot be split and the optimal transmission graph for arbitrary value of the λ parameter in (12) cannot be sum of the two graphs $T^{(k)}$ and $T^{(k+1)}$. To find the optimal transmission for any value of λ in (12) we order the transmission graphs in a sequence such that the cost of data transmission along $T^{(k)}$ is less or equal the costs along $T^{(k')}$

$$E_T(q^{(k)}) \leq E_T(q^{(k')}).$$

The minimal graph $T^{(k')}$ determines the λ_k below which the solution of (3) is given by weight matrix $q^{(k)}$ of the graph $T^{(k)}$. Similarly, the transmission graph $T^{(k')}$ for which the inequality $E_T(q^{(k+1)}) \leq E_T(q^{(k')})$ is satisfied determines the value of λ'_k above which the solution of (3) is given by $q^{(k+1)}$. We know from the solution (9), (10) and the inequalities (6), (8), that for the k -th node, for any $a \in [a_k, a_{k-1}]$ and $b \in [a_{k+1}, a_k]$, $k \in [2, N']$ between $T^{(k)}$ and $T^{(k+1)}$ there is no other optimal data transmission graphs. For this reason, from the inequality

$$E_T(q^{(k)}) \leq E_T(q^{(k+1)})$$

we get the values (14) of the parameter λ_k for which the graphs $T^{(k)}$ and $T^{(k+1)}$ are optimal.

When $a \in [1, \infty)$ and $b \in [a_1, 1]$ in (12), then the solution of (3) for a sufficiently small λ is given by the data transmission graph $T^{(0)} = \{T_{i,i-1}^{(0)}\}_{i=1}^N$, and for a sufficiently large λ by the transmission graph $T^{(1)} = \{T_{i,0}^{(1)}\}_{i=1}^N$. Increasing the parameter λ in (12) we pass by set of data transmission graphs between $T^{(0)}$ and $T^{(1)}$. The next data transmission graph, which requires more energy then the next hop transmission $T^{(0)}$ is the graph $T^{(0+)}$ in which there is an edge $T_{N,N-2}^{(0+)}$ along the N -th node transmits its Q_N of data to the $(N-2)$ node. The nodes from L_N , $i \in [1, N-1]$ uses the next hop data transmission subgraph with edges $T_{i,i-1}^{(0)}$. Note that, we cannot select an arbitrary edge $T_{i,i-2}^{(0+)}$ for $i \neq N$. This is because we want to transmit the minimal amount of data along the edge $T_{i,i-2}^{(0+)}$ and this is satisfied only for the N -th node. The total energy utilized by the network for $T^{(0+)}$ graph is given by the formula

$$E_T(q^{(0+)}) = E_T(q^{(0)}) - E_{N,N-1} - E_{N-1,N-2} + E_{N,N-2}.$$

Increasing the value of λ in (2) we must pass from the transmission graph $T^{(0)}$ to $T^{(0+)}$. Solving the inequality

$$E_T(q^{(0+)}) \leq E_T(q^{(0)})$$

with respect to the parameter λ , we get the upper bound $\lambda \leq \frac{2^a-2}{2-2^b}$ given in the lemma. When we start decrease the value of λ , above which the data transmission graph $T^{(1)}$ is optimal, then we pass to the graph $T^{(1+)}$ for which the N -th node transmits its Q_N of data along the path which consists of the two edges

$$T_{N,N-n'-1}^{(1+)}, T_{N-n'-1,0}^{(1+)} \in T^{(1+)}.$$

In other words, this is the transmission path which consists of a one hop of the length $n' + 1$ and the second hop of the length $N - (n' + 1)$. For the L_N network and $k \in [1, N']$, where N' is the number of nodes in the first part of the network, i.e., $(0, \frac{1}{2}N)$, $n' = N'$, i.e., $N' = \frac{N-2}{2}$ for N even and $N' = \frac{N-1}{2}$ N odd. The total energy consumed by the network for transmission along the graph $T^{(1+)}$ is given by the formula

$$E_0(q_N^{(1+)}) = E_0(q_N^{(1)}) - E_{N,0} + E_{N,N'+1} + E_{N'+1,0}.$$

Solving the inequality $E_T(q^{(1+)}) \leq E_T(q^{(1)})$, with respect to the parameter λ , we get the lower bound $\lambda'_0 = \frac{N^a - |N - N' - 1|^a - (N' + 1)^a}{|N - N' - 1|^b + (N' + 1)^b - N^b}$ for which the optimal data transmission graph is $T^{(1)}$. \diamond

The next two lemmas describe the optimal transmission graphs when the values of a and b in (12) does not belong to the neighboring intervals $[a_k, a_{k-1}]$.

Lemma 4. Let the exponents of the data transmission cost matrix (12) be in the intervals $a \in [a_k, a_{k-1}]$ and $b \in [a_{k'}, a_{k'-1}]$, $k \in [1, N' - 1]$, $k' \in [3, N' + 1]$, $k' \geq k + 2$, then the optimal transmission graphs to the minimum energy problem are

$$\begin{cases} T^{(k)} & \text{for } \lambda \in (0, \lambda_k], \\ T^{(k+i)} & \text{for } \lambda \in [\lambda_{k+i-1}, \lambda_{k+i}], \quad i \in [1, k' - k - 1], \\ T^{(k')} & \text{for } \lambda \in [\lambda_{k'-1}, \infty), \end{cases}$$

where λ_{k+i-1} is the solutions of the inequality

$$E_T(q^{(k-1+i)}) \leq E_T(q^{(k+i)}), \quad i \in [1, k' - k].$$

Proof. We know that, for a sufficiently small λ the solution of (3), (12), when $a \in [a_k, a_{k-1}]$ and $b \in [a_{k'}, a_{k'-1}]$, is given by the transmission graph $T^{(k)}$. From the inequality (6) and continuity of the objective function E_T follows that less optimal, the next to $T^{(k)}$ is the transmission graph $T^{(k+1)}$. The value of λ_k above which $T^{(k)}$ is not optimal is determined from the inequality $E_T(q^{(k)}) \leq E_T(q^{(k+1)})$. Similarly, for a sufficiently large λ the solution of (3) is given by the graph $T^{(k')}$. From the inequality (6) it follows that less optimal, the closest to $T^{(k')}$, is the transmission graph $T^{(k'-1)}$. Solving the inequality $E_T(q^{(k'-1)}) \leq E_T(q^{(k')})$ we get the lower bound of $\lambda_{k'-1}$ for which $T^{(k')}$ is an optimal graph. By varying the parameter λ between λ_k and $\lambda_{k'-1}$, when $a \in [a_k, a_{k-1}]$ and $b \in [a_{k'}, a_{k'-1}]$ in (12), we get the various optimal transmission graphs, different from $T^{(k)}$ and $T^{(k')}$. From the inequality (6) and continuity of the objective function E_T follows, that the only solutions of (3) for $\lambda \in [\lambda_k, \lambda_{k'}]$ can be transmission graphs $T^{(k+i)}$, $i \in [1, k' - k - 1]$. By solving the set of inequalities $E_T(q^{(k-1+i)}) \leq E_T(q^{(k+i)})$ for $i \in [1, k' - k]$ we get the ordered sequence of λ_{k+i-1} . For any λ in the interval $[\lambda_{k+i-1}, \lambda_{k+i}]$ the optimal transmission graph is $T^{(k+i)}$. \diamond

The following lemma defines the optimal transmission graph when $a \in [1, \infty)$ and $b \leq a_1$ in (12).

Lemma 5. Let the exponents of (12) be in the intervals $a \in [1, \infty)$ and $b \in [a_k, a_{k-1}]$, $k \geq 2$, then the optimal transmission graphs to the minimum energy problem are $T^{(0)}$ for $\lambda \in [0, \lambda_0]$ and $T^{(k)}$ for $k \geq 2$ $\lambda \in [\lambda_k, \infty)$, where λ_0, λ_k are given by (14).

Proof. This lemma follows from the Lemma 3. To get the upper bound λ_0 of the parameter λ for which the transmission graph $T^{(0)}$ is optimal, we need to solve the inequality $E_T(q^{(0+)}) \leq E_T(q^{(0)})$. To get the lower bound of the parameter λ for which the transmission graphs $T^{(k)}$ are optimal we have to solve the set of inequalities $E_T(q^{(k-1)}) \leq E_T(q^{(k)})$, $k \in [2, N' + 1]$ which solution λ_k are given by (14). \diamond

The optimal transmission graphs of the minimum energy problem when $a \in [a_1, 1]$ and $b \leq a_3$ in (12) are for the parameter λ .

Lemma 6. Let the exponents of (12) be in the intervals $a \in [a_1, 1]$ and $b \in [a_k, a_{k-1}]$, $k \geq 3$, then the optimal transmission graphs to the minimum energy problem are $T^{(1)}$ for $\lambda \in [0, \lambda'_0]$ and $T^{(k)}$ for $\lambda \in [\lambda_k, \infty)$, $k \in [3, N']$ where λ'_0, λ_k are given by (14).

Proof. This lemma follows from the Lemma 3. To get the upper bound λ'_0 of the parameter λ for which the transmission graph $T^{(1)}$ is optimal we need to solve the inequality $E_T(q^{(1+)}) \leq E_T(q^{(1)})$. The lower bound λ_k of the of the parameter λ for which the transmission graphs $T^{(k)}$ are optimal can be determined from the the set of inequalities $E_T(q^{(k-1)}) \leq E_T(q^{(k)})$, $k \in [3, N' + 1]$, which solution λ_k are given by (14). \diamond

4 Solution of the problem with SINR function

In previous sections we defined the minimum energy problem in terms of the data transmission cost matrix $E_{i,j}$ and data flow matrix $q_{i,j}$. In such formalism there is no information in the model about the data transmission rate, the sensors operating time and transmission errors caused by the noise and signal interference. In this section define the minimum energy problem in terms of sensors signal power, data transmission time and capacities of a transmission channels. We show, that for the optimal data transmission of the minimum energy problem in the noisy channel there is no interference of signals. For omnidirectional antennas, when the signal of the transmitting node is heard in the whole network this is equivalent to the sequential data transmission. We prove, that for the point-to-point data transmission utilized by the sensors in the physical layer, when the transmitter adjust the power of its radio signal to the distance to the receiver, the solutions of the minimum energy problem coincide with the solutions discussed in the previous sections.

We assume, that the power of the transmitting signal at the receiver must have some minimal level P_0 . This requirement means, that the transmitting node must generate the signal with the strength

$$P_{i,j} = P_0 \gamma_{i,j}^{-1}, \quad (15)$$

where $\gamma_{i,j} = \gamma(x_i, x_j)$ is the signal gain function between sender and receiver located at the points x_i and x_j of the line. For the transmission model (15) the energy consumed by the i -th sensor is given by the formula

$$E_i(t) = P_0 \sum_{j \in S_N} \gamma_{i,j}^{-1} t_{i,j}, \quad (16)$$

To get non-trivial solution of the minimum energy problem we must assume that the capacities of the transmission channels are limited, otherwise the minimum energy of each node is reached for zero transmission time $t_{i,j} = 0$. To define the size of the channel capacity we use the Shannon-Hartley formula modified by the Signal to Interference plus Noise Ratio (SINR) function, [12, 13],

$$C(x_i, x_j, U_{i,j}^n) = \log(1 + s(x_i, x_j, U_{i,j}^n)), \quad (17)$$

where

$$s(x_i, x_j, U_{i,j}^n) = \frac{P_0}{N_o + P_0 \sum_{(k,m) \in U_{i,j}^n} \gamma(x_k, x_m)^{-1} \gamma(x_k, x_j)}$$

is the SINR function and $U_{i,j}^n \subset S_N$ is some set of transmitter-receiver pairs which signal of the transmitters interfere with the signal of the i -th node. For wireless networks in which the nodes use the omnidirectional antennas and the signal is detected by any node of the network, $U_{i,j}^n$ can be defined as a set of node pairs which transmit data simultaneously, i.e.,

$$U_{i,j} = \{(i', j') \in S_N \times S_N | t_{i,j}^{(s)} = t_{i',j'}^{(s)}, t_{i,j}^{(e)} = t_{i',j'}^{(e)}\}.$$

where $t_{i,j}^{(s)}$ and $t_{i,j}^{(e)}$ is the start and the end of transmission time between i -th and j -th node. By definition $(i, j) \notin U_{i,j}$. The amount of data transmitted by the i -th node to the j -th node during the time $t_{i,j}$ with the transmission rate $c_{i,j}$ is given by the formula

$$q_{i,j} = c_{i,j} t_{i,j}. \quad (18)$$

We assume, that the transmission rate $c_{i,j}$ satisfies the inequality $0 \leq c_{i,j} \leq C(x_i, x_j, U_{i,j}^n)$, where $C(x_i, x_j, U_{i,j}^n)$ is given by (17). Because in general a set of sensors can transmit data simultaneously, thus we need to modify the node energy consumption formula (16) to the form

$$E_i(\bar{t}) = P_0 \sum_{j,n} \gamma_{i,j}^{-1} t_{i,j}^n,$$

where $\bar{t} = (t^1, \dots, t^n, \dots)$ is a tuple of time matrices t^n with elements $t_{i,j}^n$, which define the data transmission time between the i -th and j -th nodes in the presence of transmitters from the set $U_{i,j}^n$. The objective function of the minimal energy problem with SINR function is given by the formula

$$E_T(\bar{t}) = \sum_{i \in S_N} E_i(\bar{t}) = \sum_{i \in S_N} P_0 \sum_{j,n} \gamma_{i,j}^{-1} t_{i,j}^n.$$

From the data flow constraints, defined by the second formula in (3) and (18), it follows that the minimum energy is consumed by the network when the transmission rate between two nodes is maximal and equals to the channel capacity, i.e. $c_{i,j}^n = C_{i,j}^n$. Taking this into account the minimum energy problem with SINR function can be written in the form

$$\begin{cases} \min_{\bar{t}} E_T(\bar{t}), \\ \sum_{i,n} C_{i,j}^n t_{i,j}^n = Q_i + \sum_{j,n} C_{j,i}^n t_{j,i}^n, \\ t_{i,j}^n \geq 0, Q_i > 0, \end{cases} \quad (19)$$

where $C_{i,j}^n$ is given by (17). The results of the following lemma allows us further reduce the problem (19).

Lemma 7. The optimal data transmission for the minimum energy problem (19) is the transmission without interference.

Proof. For the fixed amount of data Q_i generated by each sensor, the transmission times $t_{i,j}^n$ in (19) are minimal when coefficients $C_{i,j}^n$ are maximal. From (17) it follows that maximum value of the transmission rate $C_{i,j}^n$ is achieved when $U_{i,j}^n = \emptyset$, which means that in the network there is no interference of signals. \diamond

From the Lemma 7 it follows that to solve the minimum energy problem it is enough to consider only the constant channel capacities $\forall_{i,j} C(x_i, x_j) = \log(1 + \frac{P_0}{N_o}) = C_0$. The minimum energy problem for noisy channel with the constant channel capacity can be defined by the following set of formulas

$$\begin{cases} \min_t \sum_{i \in S_N} E_i(t), \\ E_i(t) = P_0 \sum_j \gamma_{i,j}^{-1} t_{i,j}, \\ C_0 \sum_i t_{i,j} = Q_i + C_0 \sum_j t_{j,i}, \\ t_{i,j} \geq 0, Q_i > 0. \end{cases} \quad (20)$$

To solve the problem (20) for a given signal gain function $\gamma_{i,j}$ we transform (20) to the minimum energy problem defined in (3). By identifying the variables

$$\begin{cases} q_{i,j} \rightarrow P_0 t_{i,j}, \\ E_{i,j} \rightarrow \gamma_{i,j}^{-1}, \\ Q_i \rightarrow \frac{P_0}{C_0} Q_i, \end{cases}$$

we get the equivalence of the two problems. For the signal gain functions $\gamma_{i,j} = |x_i - x_j|^{-a}$, $\gamma_{i,j} = \frac{1}{|i-j|^a + \lambda|i-j|^b}$, by means of the above transformation we can obtain from (9) and (13) the solutions of (20).

5 Conclusions

In the paper we solved the minimum energy problem in one dimensional wireless sensor networks for the data transmission cost function $E(x_i, x_j) = |x_i - x_j|^a$ with any real value of the exponent a . We showed, how to find the solution of the problem when the data transmission cost function is of the form $E(x_i, x_j) = |x_i - x_j|^a + \lambda|x_i - x_j|^b$ and $a, b \in R, \lambda \geq 0$. There are several intervals for the parameter λ for which the optimal transmission graphs are not determined. For example, when

$a \in [1, \infty)$, $b \in [a_1, 1]$ in the interval $[\lambda_0, \lambda'_0]$ there are transmission graphs which lie between $T^{(0)}$ and $T^{(1)}$ and are solutions of (3). These graphs can be identified by means of the ordering method utilized in the Lemmas 3, 4, 5, 6. We defined the minimum energy problem in terms of sensors signal power, transmission time and capacities of a transmission channels. We proved, that for the point-to-point data transmission utilized by the sensors in the physical layer, when the transmitter adjust the power of its radio signal to the distance to the receiver, the solutions of the minimum energy problem written in terms of data transmission cost function $E_{i,j}$ and in terms of sensor signal power coincide.

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